Problem 9

Use the result of Problem 7(a) to find the sum of the series $\sum_{n=1}^{\infty} \arctan(2/n^2)$.

Solution

The result that was proven in Problem 7(a) is this.

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy},$$

where $xy \neq -1$. The series whose sum we have to find is the following.

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$$

Since we just have one arctan term in the summand, we'll use the formula from Problem 7(a) to break it into two terms. Thus, we set

$$\begin{aligned} x - y &= 2 \\ 1 + xy &= n^2 \end{aligned}$$

and solve for x and y with substitution. Doing this gives $x = n+1$ and $y = n-1$. By the formula from Problem 7(a), then, we have

$$\arctan \frac{2}{n^2} = \arctan(n+1) - \arctan(n-1),$$

so the series we have to evaluate becomes

$$\sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n-1)].$$

Write out the first five terms of it.

$$\begin{aligned} \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n-1)] = {} & \underbrace{\cancel{\arctan 2} - \arctan 0}_{n=1} + \underbrace{\cancel{\arctan 3} - \arctan 1}_{n=2} + \underbrace{\cancel{\arctan 4} - \cancel{\arctan 2}}_{n=3} \\ & + \underbrace{\arctan 5 - \cancel{\arctan 3}}_{n=4} + \underbrace{\arctan 6 - \cancel{\arctan 4}}_{n=5} \end{aligned}$$

Every value of n gives us two terms. The first term of n always cancels with the second term of $n+2$. Hence, this is a telescoping series, which we evaluate by calculating a limit.

$$\begin{aligned} \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n-1)] &= -\arctan 0 - \arctan 1 + \lim_{n\to\infty} \arctan n + \lim_{n\to\infty} \arctan(n+1) \\ &= 0 - \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} \\ &= \frac{3\pi}{4} \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3\pi}{4}.$$